

Almost Everywhere Convergence of Inverse Dunkl Transform on the Real Line

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Abstract

In this paper, we will first show that the maximal operator S_*^α of spherical partial sums S_R^α , associated to Dunkl transform on \mathbb{R} is bounded on $L^p\left(\mathbb{R}, |x|^{2\alpha+1} dx\right)$ functions when $\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1}$, and it implies that, for every $L^p\left(\mathbb{R}, |x|^{2\alpha+1} dx\right)$ function $f(x)$, $S_R^\alpha f(x)$ converges to $f(x)$ almost everywhere as $R \rightarrow \infty$. On the other hand we obtain a sharp version by showing that S_*^α is bounded from the Lorentz space $L^{p_i,1}\left(\mathbb{R}, |x|^{2\alpha+1}\right)$ into $L^{p_i,\infty}\left(\mathbb{R}, |x|^{2\alpha+1}\right)$, $i = 0, 1$ where $p_0 = \frac{4(\alpha+1)}{2\alpha+3}$ and $p_1 = \frac{4(\alpha+1)}{2\alpha+1}$.

Keywords: Dunkl transform, maximal function, almost everywhere convergence, Lorentz space.

1 Introduction and preliminaries

Given $\alpha \geq \frac{-1}{2}$ and a suitable function f on \mathbb{R} , its Dunkl transform D_α is defined by

$$D_\alpha f(y) = \int_{\mathbb{R}} f(x) E_\alpha(-ixy) d\mu_\alpha(x), \quad y \in \mathbb{R}; \quad (1)$$

here

$$d\mu_\alpha(x) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} |x|^{2\alpha+1} dx \quad (2)$$

$$E_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \left\{ \frac{J_\alpha(iz)}{(iz)^\alpha} + z \frac{J_{\alpha+1}(iz)}{(iz)^{\alpha+1}} \right\}, \quad (3)$$

where J_α denotes the Bessel function of the first kind of order α . The inverse Dunkl transform \check{D}_α is given by $\check{D}_\alpha f(\lambda) = D_\alpha f(-\lambda)$.

In this paper, we are interested in the almost everywhere convergence as $R \rightarrow \infty$ of the partial sums $S_R^\alpha f(x)$ where

$$S_R^\alpha f(x) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} \int_{|y| \leq R} D_\alpha f(y) E_\alpha(ixy) |y|^{2\alpha+1} dy.$$

Recall that given $\beta \geq -\frac{1}{2}$, the Hankel transform of order β of a suitable function g on $(0, \infty)$ is defined by :

$$\mathcal{H}_\beta g(y) = \int_0^\infty g(x) \frac{J_\beta(yx)}{(yx)^\beta} x^{2\beta+1} dx, \quad y > 0. \quad (4)$$

Nowak and Stempak ([3]), found an expression of the Dunkl transform D_α in terms of Hankel transform of orders α and $\alpha+1$.

Lemma 1.1 (see ([3])) Given $\alpha \geq -\frac{1}{2}$, we have :

$$D_\alpha f(y) = \mathcal{H}_\alpha(f_e)(|y|) - iy \mathcal{H}_{\alpha+1} \left(\frac{f_o(x)}{x} \right) (|y|), \quad (5)$$

where for a function f on \mathbb{R} , we denote by f_e and f_o the restrictions to $(0, \infty)$ of its even and odd parts, respectively, i.e. the functions on $(0, \infty)$ defined by

$$f_e(x) = \frac{1}{2} (f(x) + f(-x)), \quad f_o(x) = \frac{1}{2} (f(x) - f(-x)), \quad x > 0.$$

Define, the partial sums $s_R^\beta g(x)$ by :

$$s_R^\beta g(x) = \int_0^R \mathcal{H}_\beta g(y) \frac{J_\beta(xy)}{(xy)^\beta} y^{2\beta+1} dy, \quad x > 0 \quad (6)$$

and

$$s_*^\beta g(x) = \sup_{R>0} |s_R^\beta g(x)|. \quad (7)$$

In 1988, Y. Kanjin ([2]) and E. Prestini ([4]) proved, independently, the following :

Theorem 1.2 Let $\beta \geq -\frac{1}{2}$.

- If $\frac{4(\beta+1)}{2\beta+3} < p < \frac{4(\beta+1)}{2\beta+1}$ then s_*^β is bounded on $L^p((0, \infty), x^{2\beta+1})$ functions.
- If $p \leq \frac{4(\beta+1)}{2\beta+3}$ or $p \geq \frac{4(\beta+1)}{2\beta+1}$ then s_*^β is not bounded on $L^p((0, \infty), x^{2\beta+1})$ functions.

Throughout this paper we use the convention that c_α denotes a constant, depending on α and p , its value may change from line to line.

2 Almost everywhere convergence

Define linear operators $S_R^\alpha, R > 0$ and S_*^α on the Schwartz space $S(\mathbb{R})$ by

$$S_R^\alpha f(x) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} \int_{|y|\leq R} D_\alpha f(y) E_\alpha(ixy) |y|^{2\alpha+1} dy \quad (8)$$

and

$$S_*^\alpha f(x) = \sup_{R>0} |S_R^\alpha f(x)|, \quad x \in \mathbb{R}. \quad (9)$$

Lemma 2.1 Given $\alpha \geq -\frac{1}{2}$, we have

$$S_R^\alpha(f)(x) = s_R^\alpha(f_e)(|x|) + x s_R^{\alpha+1} \left(\frac{f_o(r)}{r} \right) (|x|), \quad (10)$$

$$S_*^\alpha f(x) \leq s_*^\alpha(f_e)(|x|) + |x| s_*^{\alpha+1} \left(\frac{f_o(r)}{r} \right) (|x|). \quad (11)$$

Proof. Let $x \in \mathbb{R}$. By (3), (8) and lemma 1.1, we have

$$\begin{aligned}
S_R^\alpha f(x) &= \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} \int_{|y| \leq R} \left[\mathcal{H}_\alpha(f_e)(|y|) - iy \mathcal{H}_{\alpha+1} \left(\frac{f_o(r)}{r} \right) (|y|) \right] \\
&\quad \left[2^\alpha \Gamma(\alpha+1) \left\{ \frac{J_\alpha(yx)}{(yx)^\alpha} + ixy \frac{J_{\alpha+1}(yx)}{(yx)^{\alpha+1}} \right\} |y|^{2\alpha+1} dy \right] \\
&= \frac{1}{2} \int_{|y| \leq R} \mathcal{H}_\alpha(f_e)(|y|) \frac{J_\alpha(yx)}{(yx)^\alpha} |y|^{2\alpha+1} dy \\
&\quad + \frac{ix}{2} \int_{|y| \leq R} y \mathcal{H}_\alpha(f_e)(|y|) \frac{J_{\alpha+1}(yx)}{(yx)^{\alpha+1}} |y|^{2\alpha+1} dy \\
&\quad - \frac{i}{2} \int_{|y| \leq R} y \mathcal{H}_{\alpha+1} \left(\frac{f_o(r)}{r} \right) (|y|) \frac{J_\alpha(yx)}{(yx)^\alpha} |y|^{2\alpha+1} dy \\
&\quad + \frac{x}{2} \int_{|y| \leq R} \mathcal{H}_{\alpha+1} \left(\frac{f_o(r)}{r} \right) (|y|) \frac{J_{\alpha+1}(yx)}{(yx)^{\alpha+1}} |y|^{2\alpha+3} dy
\end{aligned}$$

We note that the second and the third integrals are equal to zero. So

$$\begin{aligned}
S_R^\alpha f(x) &= \frac{1}{2} \int_{|y| \leq R} \mathcal{H}_\alpha(f_e)(|y|) \frac{J_\alpha(yx)}{(yx)^\alpha} |y|^{2\alpha+1} dy \\
&\quad + \frac{x}{2} \int_{|y| \leq R} \mathcal{H}_{\alpha+1} \left(\frac{f_o(r)}{r} \right) (|y|) \frac{J_{\alpha+1}(yx)}{(yx)^{\alpha+1}} |y|^{2\alpha+3} dy \\
&= \int_0^R \mathcal{H}_\alpha(f_e)(y) \frac{J_\alpha(|x|y)}{(|x|y)^\alpha} y^{2\alpha+1} dy + x \int_0^R \mathcal{H}_{\alpha+1} \left(\frac{f_o(r)}{r} \right) (y) \frac{J_{\alpha+1}(|x|y)}{(|x|y)^{\alpha+1}} y^{2\alpha+3} dy \\
&= s_R^\alpha(f_e)(|x|) + x s_R^{\alpha+1} \left(\frac{f_o(r)}{r} \right) (|x|).
\end{aligned}$$

Thus

$$S_*^\alpha f(x) \leq s_*^\alpha(f_e)(|x|) + |x| s_*^{\alpha+1} \left(\frac{f_o(r)}{r} \right) (|x|).$$

Proposition 2.2 Let $\alpha \geq -\frac{1}{2}$.

- If $\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1}$ then S_*^α is bounded on $L^p(\mathbb{R}, |x|^{2\alpha+1} dx)$ functions.
- If $p \leq \frac{4(\alpha+1)}{2\alpha+3}$ or $p \geq \frac{4(\alpha+1)}{2\alpha+1}$ then S_*^α is not bounded on $L^p(\mathbb{R}, |x|^{2\alpha+1} dx)$ functions.

Proof. S_*^α cannot be bounded for $p \leq \frac{4(\alpha+1)}{2\alpha+3}$ or $p \geq \frac{4(\alpha+1)}{2\alpha+1}$ (see: [2], [4]).

By theorem 1, we have for $\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1}$

$$\begin{aligned} \|s_*^\alpha(f_e)(|x|)\|_{L^p(\mathbb{R}, |x|^{2\alpha+1} dx)} &= 2 \|s_*^\alpha(f_e)\|_{L^p((0, \infty), x^{2\alpha+1} dx)} \\ &\leq c_\alpha \|f_e\|_{L^p((0, \infty), x^{2\alpha+1} dx)} \\ &\leq c_\alpha \|f\|_{L^p(\mathbb{R}, |x|^{2\alpha+1} dx)}. \end{aligned}$$

On the other hand, as in ([4], [5]), one gets

$$|x| s_*^{\alpha+1} \left(\frac{f_o(r)}{r} \right) (|x|) \leq \frac{c_\alpha}{|x|^{\alpha+\frac{1}{2}}} \left[M + H + \tilde{H} + \tilde{C} \right] \left[\frac{f_o(r)}{r} r^{\alpha+\frac{3}{2}} \right] (|x|), \quad (12)$$

where M, H, \tilde{H} and \tilde{C} denotes respectively, the maximal function, the Hilbert integral, the maximal Hilbert transform and the Carleson operator.

Let $K = M + H + \tilde{H} + \tilde{C}$ and $w \in A_p(\mathbb{R})$, $p > 1$. It is well known that

$$\|Kf\|_{L^p(\mathbb{R}, w(x)dx)} \leq c_\alpha \|f\|_{L^p(\mathbb{R}, w(x)dx)}. \quad (13)$$

Hence

$$\begin{aligned} &\left\| |x| s_*^{\alpha+1} \left(\frac{f_o(r)}{r} \right) (|x|) \right\|_{L^p(\mathbb{R}, |x|^{2\alpha+1} dx)} \\ &\leq c_\alpha \left\| |x|^{-\alpha-\frac{1}{2}} K \left[\frac{f_o(r)}{r} r^{\alpha+\frac{3}{2}} \right] (|x|) \right\|_{L^p(\mathbb{R}, |x|^{2\alpha+1} dx)} \\ &\leq c_\alpha \left\| K \left[\frac{f_o(r)}{r} r^{\alpha+3/2} \right] (|x|) \right\|_{L^p(\mathbb{R}, w(x)dx)}, \end{aligned}$$

with $w(x) = |x|^{2\alpha+1-p(\alpha+1/2)}$.

Since $\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1}$ if and only if $-1 < 2\alpha+1-p(\alpha+1/2) < p-1$,

then $w \in A_p(\mathbb{R})$ and by (13)

$$\begin{aligned} \left\| |x| s_*^{\alpha+1} \left(\frac{f_o(r)}{r} \right) (|x|) \right\|_{L^p(\mathbb{R}, |x|^{2\alpha+1} dx)} &\leq c_\alpha \left\| \frac{f_o(|x|)}{|x|} |x|^{\alpha+3/2} \right\|_{L^p(\mathbb{R}, w(x) dx)} \\ &\leq c_\alpha \|f_o(x)\|_{L^p(\mathbb{R}, |x|^{2\alpha+1} dx)} \\ &\leq c_\alpha \|f(x)\|_{L^p(\mathbb{R}, |x|^{2\alpha+1} dx)}. \end{aligned}$$

We conclude by lemma 2.1.

Corollary 2.3 For every $f \in L^p(\mathbb{R}, |x|^{2\alpha+1} dx)$, if $\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1}$, then

$$S_R^\alpha f(x) \rightarrow f(x) \quad \text{a.e. as } R \rightarrow \infty$$

3 Endpoint estimates

We recall that the Lorentz space $L^{p,q}(X, \mu)$, is the set of all measurable functions f on X satisfying

$$\|f\|_{p,q} = \left(\frac{q}{p} \int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty$$

when $1 \leq p < \infty$, $1 \leq q < \infty$, and

$$\|f\|_{p,q} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) = \sup_{\lambda>0} \lambda (d_f(\lambda))^{\frac{1}{p}} < \infty$$

when $1 \leq p \leq \infty$ and $q = \infty$. Where f^* denotes the nonincreasing rearrangement of f , i.e.

$$f^*(t) = \inf \{s > 0 / d_f(s) \leq t\}, \quad d_f(s) = \mu \{x \in X / |f(x)| > s\}.$$

In 1991 E. Romera and F. Soria [5] (see also L. Colzani and all [1]) proved the following :

Theorem 3.1 Let $\alpha > -\frac{1}{2}$, then s_*^α is bounded from the Lorentz space $L^{p_i,1}((0, \infty), x^{2\alpha+1}dx)$ into $L^{p_i,\infty}((0, \infty), x^{2\alpha+1}dx)$, $i=0,1$ when $p_0 = \frac{4(\alpha+1)}{2\alpha+3}$ and $p_1 = \frac{4(\alpha+1)}{2\alpha+1}$ is the index conjugate to p_0 .

Using this result, we will see that proposition 2.2 can be strengthened. More precisely we obtain :

Proposition 3.2 Let $\alpha > -\frac{1}{2}$, then S_*^α is bounded from the Lorentz space $L^{p_i,1}(\mathbb{R}, |x|^{2\alpha+1}dx)$ into $L^{p_i,\infty}(\mathbb{R}, |x|^{2\alpha+1}dx)$, $i = 0, 1$.

So using the formulation of Marcinkiewicz interpolation theorem in terms of Lorentz space we retrieve Proposition 2.2 ($\alpha > -\frac{1}{2}$) as a corollary.

Proof. By lemma 2.1, we have

$$\begin{aligned} \mu_\alpha \{x \in \mathbb{R} / S_*^\alpha f(x) > \lambda\} &\leq \mu_\alpha \left\{ x \in \mathbb{R} / s_*^\alpha f_e(|x|) > \frac{\lambda}{2} \right\} \\ &\quad + \mu_\alpha \left\{ x \in \mathbb{R} / |x| s_*^{\alpha+1} \left(\frac{f_o(r)}{r} \right) (|x|) > \frac{\lambda}{2} \right\}. \\ &= I + II \end{aligned}$$

By theorem 2.4, we get :

$$\begin{aligned} \mu_\alpha \left\{ x \in \mathbb{R} / s_*^\alpha f_e(|x|) > \frac{\lambda}{2} \right\} &= 2\mu_\alpha \left\{ x \in (0, \infty) / s_*^\alpha f_e(x) > \frac{\lambda}{2} \right\} \\ &\leq \frac{c_\alpha}{\lambda^{p_i}} \|f_e\|_{p_i,1} \leq \frac{c_\alpha}{\lambda^{p_i}} \|f\|_{p_i,1}. \end{aligned}$$

To estimate II, we follow closely [5] and we sketch a proof for completeness. We decompose the set :

$$\begin{aligned} &\left\{ x \in \mathbb{R} / |x| s_*^{\alpha+1} \left(\frac{f_o(r)}{r} \right) (|x|) > \frac{\lambda}{2} \right\} \\ &= \bigcup_{k \in \mathbb{Z}} \left\{ x \in \mathbb{R} / |x| \in I_k, |x| s_*^{\alpha+1} \left(\frac{f_o(r)}{r} \right) (|x|) > \frac{\lambda}{2} \right\}, \end{aligned}$$

where $I_k = [2^k, 2^{k+1}[$.

Put $g(r) := \frac{f_o(r)}{r} = g_k^1(r) + g_k^2(r)$, with $g_k^1 = g\chi_{I_k^*}$, $g_k^2 = g\chi_{(I_k^*)^c}$, where $I_k^* =]2^{k-1}, 2^{k+2}[$.

By (12), we have :

$$|x| s_*^{\alpha+1} (g_k^1(r)) (|x|) \leq \frac{c_\alpha}{|x|^{\alpha+1/2}} K (g_k^1(r) r^{\alpha+3/2}) (|x|).$$

By ([5], p: 1021), we have for $1 < p < \infty$,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \mu_\alpha \left\{ x \in \mathbb{R} / |x| \in I_k, \frac{1}{|x|^{\alpha+1/2}} K (g_k^1(r) r^{\alpha+3/2}) (|x|) > \frac{\lambda}{2} \right\} \\ \leq \frac{c_\alpha}{\lambda^p} \|f_o\|_{L^p(\mathbb{R}, |x|^{2\alpha+1} dx)}^p \leq \frac{c_\alpha}{\lambda^p} \|f\|_{L^p(\mathbb{R}, |x|^{2\alpha+1} dx)}^p \leq \frac{c_\alpha}{\lambda^p} \|f\|_{p,1}^p. \end{aligned}$$

On the other hand as in ([5], p: 1021), we have

$$\begin{aligned} |x| s_*^{\alpha+1} (g_k^2(r)) (|x|) &\leq \frac{c_\alpha}{|x|^{\alpha+1/2}} \int_0^\infty \frac{s^{\alpha+3/2} |f_o(s)|}{s(|x| + s)} ds \\ &\leq \frac{c_\alpha}{|x|^{\alpha+3/2}} \int_0^\infty |f_o(s)| s^{\alpha+1/2} ds \\ &\leq \frac{c_\alpha}{|x|^{\alpha+3/2}} \int_{\mathbb{R}} |f_o(s)| \frac{1}{|s|^{\alpha+1/2}} |s|^{2\alpha+1} ds. \end{aligned}$$

Remark that we have considered f_o as a function defined on \mathbb{R} .

As the same we get

$$\begin{aligned} |x| s_*^{\alpha+1} (g_k^2(r)) (|x|) &\leq \frac{c_\alpha}{|x|^{\alpha+1/2}} \int_0^\infty |f_o(s)| s^{\alpha-1/2} ds \\ &\leq \frac{c_\alpha}{2|x|^{\alpha+1/2}} \int_{\mathbb{R}} |f_o(s)| \frac{1}{|s|^{\alpha+3/2}} |s|^{2\alpha+1} ds. \end{aligned}$$

Using the following facts :

$$\frac{1}{|x|^{\alpha+\frac{1}{2}}} \in L^{p_1, \infty} (\mathbb{R}, |x|^{2\alpha+1}),$$

$$\frac{1}{|x|^{\alpha+\frac{3}{2}}} \in L^{p_0, \infty}(\mathbb{R}, |x|^{2\alpha+1}),$$

and Holder's inequality for the Lorentz spaces, we arrive to :

$$\mu_\alpha \left\{ x \in \mathbb{R} / |x| s_*^{\alpha+1} (g_k^2(r)) (|x|) > \frac{\lambda}{2} \right\} \leq \frac{c_\alpha}{\lambda^{p_i}} \|f_o\|_{p_i,1}^{p_i} \leq \frac{c_\alpha}{\lambda^{p_i}} \|f\|_{p_i,1}^{p_i},$$

which completes the proof.

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